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Common Fixed Point Theorem in Fuzzy 2-Metric Space Using Implicit Relation

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ABSTRACT: In this paper, we prove some common fixed point theorems for weak commuting mappings in fuzzy 2-metric space without continuity condition.**

I. INTRODUCTION

The idea of fuzzy metric space was introduced by Kramosil and Michalek [4] which was, later on, modified by George and Veeramani [2]. Many authors like Jungck and Rohades [3], Vasuki [12], Singh and Jain [11] used the concept of R-weakly commuting mappings and weak-compatible mappings to prove fixed point theorems in fuzzy metric spaces. The notion of weakly commuting introduced by Seesa [9] was improved by Pathak [5] by defining weak* commuting and weak**commuting mappings in metric spaces and prove some fixed point theorem.

Recently Singh and Jain [11], prove a unique common fixed point theorem in fuzzy metric space for four mappings with implicit relations including continuity condition. In this paper, with the concepts of weak** commuting, we prove some common fixed point theorems in fuzzy 2-metric space by using implicit relation without continuity condition.

II. PRELIMINARIES

In this section we start some definitions and known results which will be used in the proof of main result.

Definition 2.1. The 3-tuple (X, M,*) is called a fuzzy 2-metric space if X is an arbitrary non-empty set,* is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $t > 0$:

(FM-1) $M(x, y, z, t) > 0$,

(FM-2) $M(x, y, z, t) = 1$ iff at least two out of three Points are equal,

(FM-3) $M(x, y z, t) = M(x, z, y, t) = M(y, z, x, t)$

(FM-4) $M(x, y, z, t_1+t_2+t_3) \geq M(x, y, u, t_1)*M(x, u, z, t_2) *M(u, y, z, t_3)$, for $t_i > 0$

(FM-5) $M(x, y, z \cdot): (0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, z, t) = 1$.

Definition 2.2. A pair (A, S) of self maps of a fuzzy 2-metric space (X, M, *) is said to be weak** commuting if $A(X) \subset S(X)$ and for all $x \in X$,

$$\begin{aligned} M(S^2A^2v, A^2S^2v, z, t) &\geq M(S^2Av, A^2Sv, z, t) \\ &\geq M(SA^2v, AS^2v, z, t) \\ &\geq M(SAv, ASv, z, t) \\ &\geq M(S^2v, A^2v, z, t). \end{aligned}$$

If $A^2 = A$ and $S^2 = S$ then the weak** commutative reduces to weak commutative.

2.3. A class of implicit function: Let Φ be the set of all real continuous functions $F:(R^+)^5 \to R$, non decreasing in the first argument satisfying the following conditions:

(a) For $u, v \geq 0$, $F(u, v, u, v, 1) \geq 0$ implies that $u \geq v$.

(b) $F(u,1,1,u,1) \geq 0$ or $F(u,u,u,1,u) \geq 0$ or $F(u,1,u,1,u) \geq 0$ implies that $u \geq 1$.

Example:
Define $F(t_1, t_2, t_3, t_4, t_5) = 20 t_1 - 18 t_2 + 10 t_3 - 12 t_4 - t_5 + 1$.

Lemma 2.4. If for all $x, y \in X$, $t > 0$ and $0 < k < 1$, $M(x, y, kt) \geq M(x, y, t)$, then $x = y$.

Lemma 2.5 Let $\{y_n\}$ be a sequence in fuzzy 2-metric space (X, M, *) with the condition FM-6.If there exists $k \in (0,1)$ such that $M(y_n, y_{n+1}, z, kt) \geq M(y_n, y_{n+1}, z, t)$, for all $t > 0$ and $n \in N$, then $\{y_n\}$ is a Cauchy sequence in X.

III. MAIN RESULTS

Theorem 3. Let A, B, T and S be self mappings on a complete fuzzy 2-metric space (X, M, *) satisfying the following

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) The pairs (A, S) and (B, T) are weak** commutative.

(iii) T(X) and S(X) are complete

(iv) For some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y, z \in X$ and $t > 0$,

$$F\{M(A^2x, B^2y, z, kt), M(T^2y, S^2x, z, t), M(A^2x, T^2y, z, t), M(B^2y, T^2y, z, kt), M(T^2y, A^2x, z, t)\} \geq 0$$

Then A, B, T and S have a unique common fixed point in X.

Proof: Let $x_0 \in X$. By (i),there exist $x_1, x_2 \in X$ such that

$$A^2x_1 = T^2x_1 \text{ and } B^2x_2 = S^2x_2$$

In this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for =1, 2, 3, …

$$y_{2n+1} = A^2x_{2n} = T^2x_{2n+1},$$
$$y_{2n+2} = B^2x_{2n+1} = S^2x_{2n+2}$$

Now using condition (iv-a) for any $z \in X$, with $x = x_{2n}$, $y = x_{2n+1}$, we get,

$$F\{M(A^2x_{2n}, B^2x_{2n+1}, z, kt), M(T^2x_{2n+1}, S^2x_{2n}, z, t), M(A^2x_{2n}, T^2x_{2n+1}, z, t), M(B^2x_{2n+1}, T^2x_{2n+1}, z, kt), M(T^2x_{2n+1}, A^2x_{2n}, z, t)\} \geq 0,$$

That is,

$$F\{M(y_{2n+1}, y_{2n+2}, z, kt), M(y_{2n+1}, y_{2n}, z, t), M(y_{2n+1}, y_{2n+1}, z, t), M(y_{2n+2}, y_{2n+1}, z, kt), M(y_{2n+1}, y_{2n+1}, z, t)\} \geq 0.$$

By condition 2.3(a),

$$M(y_{2n+2}, y_{2n+1}, z, kt) \geq M(y_{2n+1}, y_{2n}, z, t)$$

Thus by lemma 2.5, $\{y_n\}$ is a Cauchy sequence. By completeness of X, there exists p in X such that $\{y_n\}$ converges to p. Hence its subsequence's $\{A^2x_{2n}\}$, $\{B^2x_{2n+1}\}$, $\{T^2x_{2n+1}\}$ and $\{S^2x_{2n+2}\}$ also converge to p.

By (iii) T(X) is complete, therefore, $p \in T(X)$, thus there exists $u \in X$ such that $p = T^2u$.

Now put, $x = x_{2n}$ and $y = u$ in (iv), we get for every $z \in X$,

$$F\{M(A^2x_{2n}, B^2u, z, kt), M(T^2u, S^2x_{2n}, z, t), M(A^2x_{2n}, T^2u, z, t), M(B^2u, T^2u, z, kt), M(T^2u, A^2x_{2n}, z, t)\} \geq 0,$$

Taking limit $n \to \infty$, we get

$$F\{M(p, B^2u, z, kt), M(p, p, z, t), M(p, p, z, t), M(B^2u, p, z, kt), M(p, p, z, t)\} \geq 0,$$

that is,

$$F\{M(p, B^2u, z, kt), 1, 1, M(B^2u, p, z, kt), 1\} \geq 0,$$

So that by 2.3(b),

$$M(B^2u, p, z, kt), 1\} \geq 1$$

Hence, $p = B^2u = T^2u$. …(3.1)

Using weak** commutatively of the pair (T, B), we have

$$M(T^2B^2u, B^2T^2u, z, t) \geq M(T^2Bu, B^2Tu, z, t) \geq M(TB^2u, BT^2u, z, t) \geq M(TBu, BTu, z, t) \geq M(T^2u, B^2u, z, t)$$

Hence, $T^2B^2u = T^2S^2u$, therefore by (3.1)

$$T^2p = B^2p. \quad ...(3.2)$$

Now we put $x = x_{2n}$ and $y = p$ in (iv), we get for every $z \in X$,

$$F\{M(A^2x_{2n}, B^2p, z, kt), M(T^2p, S^2x_{2n}, z, t), M(A^2x_{2n}, T^2p, z, t), M(B^2p, T^2p, z, kt), M(T^2p, A^2x_{2n}, z, t)\} \geq 0,$$

Taking limit $n \to \infty$, we get

$$F\{M(p, B^2p, z, kt), M(T^2p, p, z, t), M(p, T^2p, z, t), M(B^2p, T^2p, z, kt), M(T^2p, p, z, t)\} \geq 0,$$

This gives by using (3.2)

$$F\{M(p, B^2p, z, kt), M(B^2p, p, z, t), M(p, B^2p, z, t), 1, M(B^2p, p, z, t)\} \geq 0.$$

Since F is non decreasing in first argument, we have

$$F\{M(p, B^2p, z, t), M(B^2p, p, z, t), M(p, B^2p, z, t), 1, M(B^2p, p, z, t)\} \geq 0.$$

Therefore by 7-2-3(b), we have

$$M(p, B^2p, z, t) \geq 1$$

Hence, $p = B^2p = T^2p$ …(3.3)

Since, $B(X) \subset S(X)$, there exists $v \in X$ such that $p = B^2p = S^2v$.

Put $x = v$ and $y = p$ in (iv), we get

$$F\{M(A^2v, B^2p, z, kt), M(T^2p, S^2v, z, t), M(A^2v, T^2p, z, t), M(B^2p, T^2p, z, kt), M(T^2p, A^2p, z, t)\} \geq 0,$$

That is,

$$F\{M(A^2v, p, z, kt), M(p, p, z, t),$$

$M(A^2v, p, z, t)$,
$M(p, p, z, kt)$,
$M(p, A^2p, z, t)\} \geq 0$,

Since F is non decreasing in first argument, we have

$$F\{M(A^2v, p, z, t), 1, M(A^2v, p, z, t), 1, M(p, A^2p, z, t)\} \geq 0,$$

Therefore by 7-2-3(b), we have

$$M(p, A^2v, z, t) \geq 1$$

Hence, $p = A^2v = S^2v$. …(3.4)

Using weak** commutatively of the pair (S, A), we have

$$M(S^2A^2v, A^2S^2v, z, t) \geq M(S^2Av, A^2Sv, z, t) \geq M(SA^2v, AS^2v, z, t) \geq M(SAv, ASv, z, t) \geq M(S^2v, A^2v, z, t)$$

Hence, $S^2A^2v = A^2S^2v$, therefore by (3.4)

$$S^2p = A^2p \quad …(3.5)$$

Now we put $x = p$ and $y = x_{2n+1}$ in (iv), we get for every $z \in X$,

$$F\{M(A^2p, B^2x_{2n+1}, z, kt), M(T^2x_{2n+1}, S^2p, z, t), M(A^2p, T^2x_{2n+1}, z, t), M(B^2x_{2n+1}, T^2x_{2n+1}, z, kt), M(T^2x_{2n+1}, A^2p, z, t)\} \geq 0,$$

Taking limit $n\to\infty$, we get

$$F\{M(A^2p, p, z, kt), M(p, S^2p, z, t), M(A^2p, p, z, t), M(p, p, z, kt), M(p, A^2p, z, t)\} \geq 0,$$

Using (3.5), we have

$$F\{M(A^2p, p, z, kt), M(p, A^2p, z, t), M(A^2p, p, z, t), 1, M(p, A^2p, z, t)\} \geq 0,$$

Since F is non decreasing in first argument, we have

$$F\{M(A^2p, p, z, t), M(p, A^2p, z, t), M(A^2p, p, z, t), 1, M(p, A^2p, z, t)\} \geq 0,$$

Therefore by 7-2-3(b), we have

$$M(p, A^2p, z, t) \geq 1$$

Hence, $p = A^2p = S^2p$,

Therefore by (3.3),

$$p = A^2p = B^2p = S^2p = T^2p \quad …(3.6)$$

Now put $x = Ap$ and $y = p$ in (iv), we get

$$F\{M(A^2Ap, B^2p, z, kt), M(T^2p, S^2Ap, z, t), M(A^2Ap, T^2p, z, t), M(B^2p, T^2p, z, kt), M(T^2p, A^2Ap, z, t)\} \geq 0,$$

Since (A, S) is weak** commutative, therefore $S^2Ap = AS^2p$, we have using (3.6)

$$F\{M(Ap, p, z, kt), M(p, Ap, z, t), M(Ap, p, z, t), M(p, p, z, kt), M(p, Ap, z, t)\} \geq 0,$$

That is

$$F\{M(Ap, p, z, kt), M(p, Ap, z, t), M(Ap, p, z, t), 1, M(p, Ap, z, t)\} \geq 0,$$

Since F is non decreasing in first argument, we have

$$F\{M(Ap, p, z, t), M(p, Ap, z, t), M(Ap, p, z, t), 1, M(p, Ap, z, t)\} \geq 0,$$

Therefore by 2.3(b), we have

$$M(p, Ap, z, t)\} \geq 1$$

Hence, $p = Ap$. Similarly we can show that $p = Bp$, $p = Tp$, $p = Sp$. This shows that p is common fixed point of A, B, T and S.

Uniqueness: Let p and q be two common fixed points of A, B, T and S.

Put $x = p$ and $y = q$ in (iv) we get

$$F\{M(A^2p, B^2q, z, kt), M(T^2q, S^2p, z, t), M(A^2p, T^2q, z, t), M(B^2q, T^2q, z, kt), M(T^2q, A^2p, z, t)\} \geq 0,$$

Using (3.6) we have

$$F\{M(p, q, z, kt), M(q, p, z, t), M(p, q, z, t), M(q, q, z, kt), M(q, p, z, t)\} \geq 0,$$

That is

$$F\{M(p, q, z, kt), M(q, p, z, t), M(p, q, z, t), 1, M(q, p, z, t)\} \geq 0,$$

Since F is non decreasing in first argument, we have

$$F\{M(p, q, z, t), M(q, p, z, t), M(p, q, z, t), 1, M(q, p, z, t)\} \geq 0,$$

Therefore

$$M(q, p, z, t)\} \geq 0$$

Hence, $p = q$. This proves that A, B, T and S have unique common fixed point.

By taking $A = UV$ and $B = WP$ in the proof of theorem 3, we have the following result.

Theorem 4. Let U, V, W, P, T and S be self mappings on a complete fuzzy 2-metric space (X, M, *) satisfying the following

(i) $UV(X) \subset T(X)$, $WP(X) \subset S(X)$,
(ii) The pairs (UV,S) and (WP,T) are weak** commutative.
(iii) T(X) and S(X) are complete.

(iv) For some $F\in\Phi$, there exists $k \in (0,1)$ such that for all x,

$$y, z \in X \text{ and } t > 0,$$
$$F\{M(U^2V^2x, W^2P^2y, z, kt),$$
$$M(T^2y, S^2x, z, t),$$
$$M(U^2V^2x, T^2y, z, t),$$
$$M(W^2P^2y, T^2y, z, kt),$$
$$M(T^2y, U^2V^2x, z, t)\} \geq 0$$

Then U, V, W, P, T and S have a unique common fixed point in X.

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